3. DATA TRANSMISSION

During the transmission of data over the channel, it is corrupted by noise. Hence at the receiver, the noisy signal is received. Therefore correct detection of the transmitted signal is difficult. For example consider the transmitted signal and received noisy signal as shown in Fig 4.1(a) and (b)

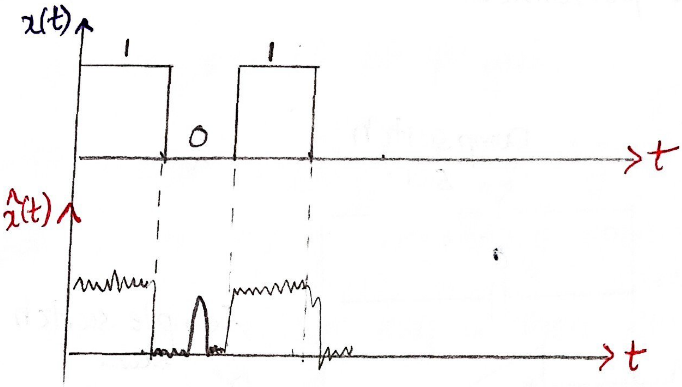


Fig 1: Effect of noise at the receiver can make wrong decision

(a) The signal transmitted at the transmitter

(b) The received noisy signal at the receiver

The received signal $\hat{x}(t)$ is a noisy signal at the receiver. Let us consider that, the detector checks $\hat{x}(t)$ at 'T' during every bit interval.

In the above figure observe that the decision in first interval will be correct i.e symbol 1. But in second interval, the decision will be '1' but it is wrong.

At the time when detector checks $\hat{x}(t)$, noise pulse is detected and decision is taken in favour of '1'. But actually symbol '0' is transmitted in second interval as shown in Fig 1(a). Thus errors are introduced because of noise.

The detecting method of the baseband signal perform following jobs:

1. The detection method should attenuate noise and amplify signal, i.e it should improve signal to noise ratio of the received signal.

2. The detection method should check the received signal at the time instant in the bit interval when signal to Noise ratio is maximum

3. The detection should be performed with minimum error probability.

→ **Baseband Signal Receiver**

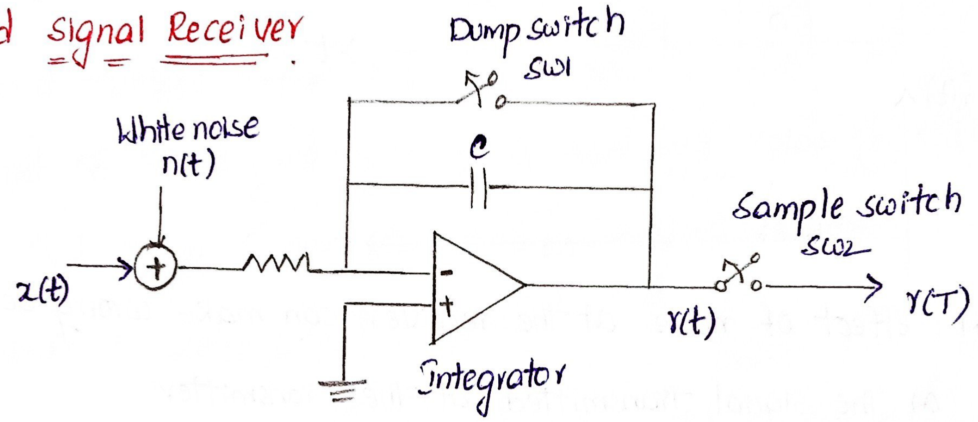


Fig 2: Integrate and dump filter

Consider a very simple and basic detector for the detection of digital signals. Fig 2: shows the circuit diagram of such integrate and dump filter.

The digital signal $x(t)$ is corrupted by white noise $n(t)$ during transmission over channel. Such noisy signal $[x(t)+n(t)]$ is given to the input of Integrate and Dump Filter.

The capacitor is discharged fully at the beginning of the bit interval. This is achieved by temporarily closing switch SW1 at the beginning of the bit interval. The integrator then integrates noisy input signal over one bit period. This integrated signal is shown as $r(t)$ in Fig 2.

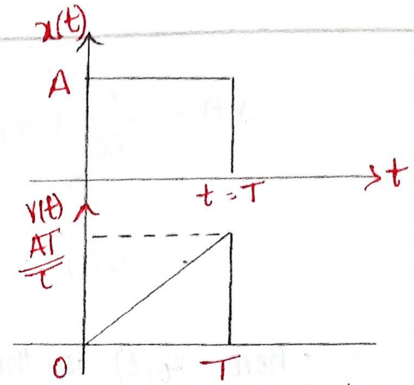


Fig 3 (a) : Input pulse to the integrator
3 (b) : output of integrator.

For the square pulse input, the output of the ~~tri~~ integrator will be a triangular pulse as shown in Fig 3.

In Fig 3(b) shows the waveform of r(t). At the end of bit period i.e $t = T$, the value of r(t) reaches to its maximum amplitude. Therefore the value of r(t) is sampled at the end of bit period.

We will further prove that the signal to Noise ratio is maximum at the end of bit period. Depending upon the value of $r(T)$, the decision is taken. The dump switch SW_1 is then closed momentarily to discharge the capacitor to receive next bit.

Thus integrator integrates or generates output independent of the value of the previous bit. This shows that the detection in integrate and dump filter is unaffected by values of previous bits. In Fig 3(b), the output of integrator will decrease after $t > T$.

→ Signal to Noise Ratio of the Integrator and Dump Filter

The output of the integrator can be written as

$$r(t) = \frac{1}{RC}\int_0^T [x(t) + n(t)]\, dt \quad \text{— (1)}$$

Here the integration is performed over one bit period i.e from 0 to T.

The noisy signal $x(t) + n(t)$ is input to an integrator. We can write the above equation as

$$y(t) = \frac{1}{RC}\int_0^T x(t)\,dt + \frac{1}{RC}\int_0^T n(t)\,dt$$

$$= x_0(t) + n_0(t) \quad — (2)$$

where $x_0(t)$ is the output signal voltage and $n_0(t)$ is the output noise.

Consider the output signal voltage,

$$x_0(t) = \frac{1}{RC}\int_0^T x(t)\,dt$$

Since the value of $x(t) = A$ (fixed) from 0 to T, we can write above above equation as

$$x_0(t) = \frac{1}{RC}\int_0^T A\,dt = \frac{A}{RC}[t]_0^T$$

$$= \frac{AT}{RC}$$

$$x_0(t) = \frac{AT}{\tau} \quad -(3) \quad [\because \text{Time constant } RC = \tau]$$

The normalized signal power in standard 1Ω resistance, will be

$$\text{Output signal power} = \frac{x_0^2(t)}{1\Omega} = \frac{A^2T^2}{\tau^2} \quad -(4)$$

Now let us calculate the noise power. But before that we have to evaluate the transfer function of the integrator. A network which performs integration operation has the transfer function of $\frac{1}{j\omega RC}$.

A delay of $t = T$ in time domain is equivalent to $e^{-j\omega T}$ in frequency domain. Thus the network performing integration over the period of T can be represented by the following transfer function

$$H(f) = \frac{1-e^{-j\omega T}}{j\omega RC} \quad \text{--- (5)}$$

Substitute $\omega = 2\pi f$ and $RC = \tau$ in above equation as,

$$H(f) = \frac{1-e^{-j2\pi f T}}{j\omega \tau}$$

$$= \frac{1-[\cos 2\pi f T - j\sin(2\pi f T)]}{j2\pi f \tau}$$

$$= \frac{1-\cos(2\pi f T)}{(j2\pi f \tau)} + \frac{\sin(2\pi f T)}{2\pi f \tau}$$

$$= \frac{\sin(2\pi f T)}{2\pi f \tau} - j\frac{(1-\cos(2\pi f T))}{2\pi f \tau}$$

The magnitude of this transfer function will be

$$|H(f)|^2 = \frac{\sin^2(2\pi f T) + (1-\cos(2\pi f T))^2}{(2\pi f \tau)^2}$$

$$= \frac{\sin^2(2\pi f T) + 1 + \cos^2(2\pi f T) - 2\cos(2\pi f T)}{(2\pi f \tau)^2}$$

$$= \frac{2-2\cos(2\pi f T)}{(2\pi f \tau)^2} \qquad [\because \sin^2(2\pi f T) + \cos^2(2\pi f T) = 1]$$

$$= \frac{2(1-\cos(2\pi f T))}{(2\pi f T)^2} \qquad [1-\cos 2\theta = 2\sin^2\theta]$$

$$= \frac{2.2\sin^2(\pi f T)}{(2\pi f \tau)^2}$$

$$|H(f)|^2 = \frac{\sin^2(\pi f T)}{(\pi f \tau)^2} \quad \text{--- (6)}$$

The average power of the output noise signal $n_o(t)$ is obtained by integrating its power density spectrum i.e

By definity Power, $P = \int_{-\infty}^{\infty} s(f)\, df$ — (7)

In standard 1Ω resistance, the noise power will be $\frac{\overline{n_o^2(t)}}{1\Omega} = \overline{n_o^2(t)}$

Here mean square value of noise is taken since it is random signal i.e

Noise power, $\overline{n_o^2(t)} = \int_{-\infty}^{\infty} S_{no}(f)\, df$ — (8)

The input and output power spectral densities are related as

$$|H(f)|^2 = \frac{S_{no}(f)}{S_{ni}(f)} \quad -(9)$$

where $|H(f)$ is transfer function of filter

$S_{no}(f)$ is psd of output noise

$S_{ni}(f)$ is psd of input noise

We are assuming that white noise is present. The power spectral density (psd) of this noise is

$$S_{ni}(f) = \frac{N_o}{2} \quad -(10)$$

substitute the eq (10) in (9), we get

$$S_{no}(f) = |H(f)|^2 \cdot \frac{N_o}{2} \quad -(11)$$

Substitute the eq (11) in eq (8), then we get

$$\overline{n_o^2(t)} = \int_{-\infty}^{\infty} |H(f)|^2 \cdot \frac{N_o}{2}\, df \quad -(12)$$

putting the value of $|H(f)|^2$ in eq (15), we get

$$\overline{n_o^2(t)} = \int_{-\infty}^{\infty} \frac{\sin^2(\pi f T)}{(\pi f \tau)^2} \cdot \frac{N_0}{2} \cdot df$$

Let $\pi f \tau = x$ and $\pi f \tau = x$

$df = \frac{dx}{\pi \tau}$ $\quad$ $\pi f = \frac{x}{\tau}$

$\pi f T = \frac{xT}{\tau}$

$$\therefore \quad \overline{n_o^2(t)} = \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{xT}{\tau}\right)}{x^2} \cdot \frac{N_0}{2} \cdot \frac{dx}{\pi\tau}$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{xT}{\tau}\right)}{x^2\left(\frac{T}{\tau}\right)^2} \cdot \left(\frac{T}{\tau}\right)^2 \cdot \frac{1}{\pi\tau} \cdot dx$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{xT}{\tau}\right)}{x \cdot \pi f \tau \cdot \left(\frac{T}{\tau}\right)^2} \cdot \left(\frac{T}{\tau}\right)^2 \cdot \frac{1}{\pi\tau} \cdot dx$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{xT}{\tau}\right)}{x \cdot \pi f \cancel{\tau} \cdot \frac{\cancel{T}}{\cancel{\tau}} \cdot \frac{T}{\tau}} \cdot \left(\frac{T}{\tau}\right)^2 \cdot \frac{1}{\pi\tau} \, dx$$

$$= \frac{N_0 T^2}{2\pi^2\tau^3}\int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{xT}{\tau}\right)}{\left(\frac{xT}{\tau}\right)} \cdot dx$$

Let $\frac{xT}{\tau} = u$

$dx \cdot \frac{T}{\tau} = du \Rightarrow dx = du \cdot \frac{\tau}{T}$

Limits will be unchanged, therefore the above equation becomes

$$\overline{n_o^2(t)} = \frac{N_0}{2} \cdot \frac{T^2}{\pi^2 \tau^3} \int_{-\infty}^{\infty} \frac{\sin^2 u}{u} \cdot \frac{\tau}{T}\, du$$

$$= \frac{N_0}{2} \cdot \frac{T^{\cancel{2}}}{\pi^2 \tau^3} \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} \cdot u \cdot \frac{\cancel{\tau}}{\cancel{T}}\, du$$

$$= \frac{N_0 T}{2\pi^2 \tau^2} \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} \cdot \frac{\pi T}{\tau} \cdot du \qquad \left[\because u = \frac{\pi T}{\tau}\right]$$

$$= \frac{N_0 T}{2\pi^{\cancel{2}} \tau^2} \int_{-\infty}^{\infty} \left(\frac{\sin u}{u}\right)^2 \cdot \frac{\cancel{\pi}\cancel{T} \cdot \cancel{\tau}}{\cancel{\tau}} \cdot du$$

$$= \frac{N_0 T}{2\pi\tau^2} \int_{-\infty}^{\infty} \left(\frac{\sin u}{u}\right)^2 \cdot du$$

Since the function $\frac{\sin u}{u}$ is squared, we can write above equation as,

$$\overline{n_o^2(t)} = \frac{N_0 T}{2\pi\tau^2} \cdot 2 \int_{0}^{\infty} \left(\frac{\sin u}{u}\right)^2 \cdot du$$

$$= \frac{N_0 T}{2\cancel{\pi}\tau^2} \cdot \cancel{2} \cdot \frac{\cancel{\pi}}{\cancel{2}}$$

$$\overline{n_o^2(t)} = \frac{N_0 T}{2\tau^2} \quad \text{— (13)}$$

$\therefore$ The Noise power at the output is $\overline{n_o^2(t)} = \frac{N_0 T}{2\tau^2}$

The signal to noise power ratio at the output of integrator as

$$(SNR)_0 = \frac{\text{output signal power}}{\text{output Noise power}} = \frac{\frac{A^2 T^{\cancel{2}}}{\cancel{\tau^2}}}{\frac{N_0 \cancel{T}}{2\cancel{\tau^2}}} = \frac{A^2 T}{\left(\frac{N_0}{2}\right)}$$

Thus, signal to Noise ratio of Integrate and Dump receiver

$$\boxed{\rho = \frac{A^2T}{N_0/2}} \quad -(14)$$

<u>Comments about signal to Noise ratio</u> :

1. The above result shows that signal to noise ratio improves in proportion to sampling period 'T'. It also increases as signal amplitude 'A' is more.

2. Since noise has gaussian distribution and zero mean value at any time, the output of integrator also increases by very small amount at the end of the interval.

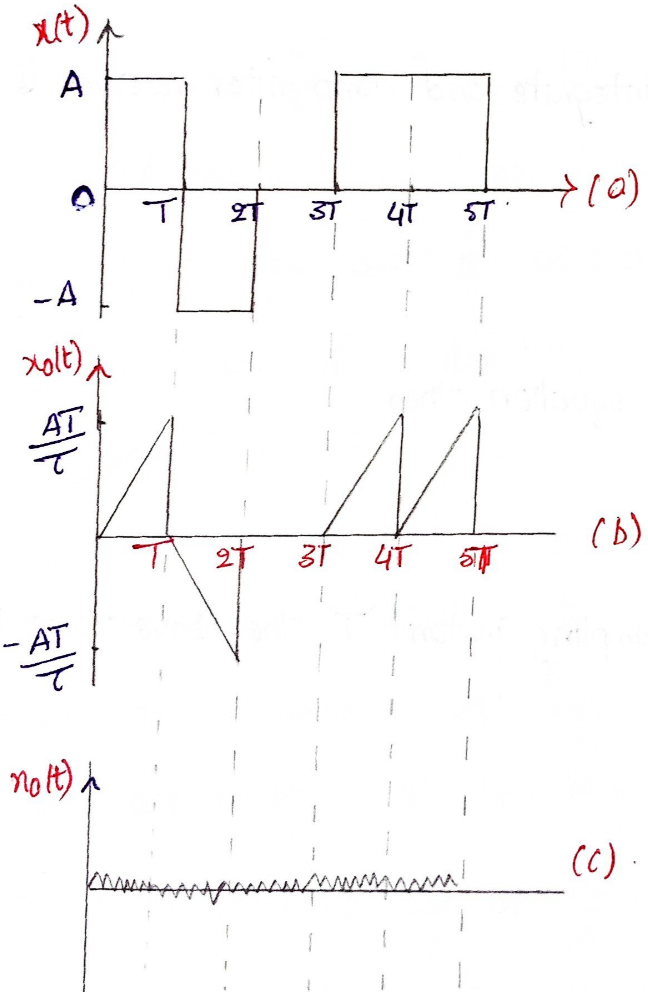


Fig 4 : shows the waveforms of integrate and dump filter receiver for the rectangular pulses input. observe that the output signal voltage reaches to the value of $\pm \frac{AT}{\tau}$ at the sampling instant. This is the maximum signal voltage. But the noise voltage $n_0(t)$ does not increase in same proportion. This is because the noise has zero average value for "Gaussian distribution".

Fig 4(a) : Input signal to the integrate and dump filter (receiver)
(b) : output signal of the integrate and dump filter
(c) : output noise of the integrate and dump receiver

→ Prove that the maximum signal to Noise ratio of the integrate and dump filter receiver is given as

$$\rho_{max} = \frac{2E}{N_0}$$

When the input signal $x(t)$ is rectangular pulses of amplitudes $\pm A$ and duration T.

Sol We know that energy of the signal $x(t)$ is given by standard relation as

$$E = \int_{-\infty}^{\infty} x^2(t)\,dt$$

Since $x(t)$ is a rectangular pulse of amplitude $\pm A$ and duration T, we can write above equation as

$$E = \int_0^T (\pm A)^2 dt = A^2 T$$

The signal to noise ratio of integrate and dump filter receiver is given by eq (14)

$$\rho = \frac{A^2 T}{N_0/2}$$

putting $A^2T = E$ in the above equation, then

$$\rho = \frac{E}{(N_0/2)} = \frac{2E}{N_0}$$

Since output is maximum at sampling instant T, the above value is maximum i.e.,

$$\boxed{\rho_{max} = \frac{2E}{N_0}}$$

→ Probability of Error in Integrate and Dump filter Receiver

The Noise in the signal leads to wrong decision at the receiver. Probability of error P_e is the good measure for performance of the detector. The output of the integrator is given as

$$y(t) = x_o(t) + n_o(t) \quad — (1)$$

For the positive pulse of amplitude A, $x_o(t) = \frac{AT}{\tau}$

Similarly for the inputs pulse of amplitude $-A$, $x_o(t) = \frac{-AT}{\tau}$

Therefore, we can write output $y(t)$ as

$$y(t) = \frac{AT}{\tau} + n_o(t) \quad \text{for } x(t) = A \quad — (2)$$

Illy $$y(t) = \frac{-AT}{\tau} + n_o(t) \quad \text{for } x(t) = -A \quad — (3)$$

Consider that $x(t) = -A$. Then if noise $n_o(t)$ is greater than $\frac{AT}{\tau}$, output $y(t)$ will be positive according to eq (3). Then the receiver will decide in favour of symbol $+A$, which is wrong decision. Thus error is introduced.

Similarly consider that $x(t) = +A$. Then if noise $n_o(t) < \frac{-AT}{\tau}$, output $y(t)$ will be negative according to equation (2). This leads to decision in favour of $-A$ which is erroneous. Based on the above discussion, Comments are made in the following table about probability of error.

These probabilities can be obtained from PDF of $n_o(t)$. We know that the probability density function (PDF) of the gaussian distributed function is given by standard relation as

Input $x(t)$	Value of $n_0(t)$ for error in the output	Probability of error
$-A$	Error introduced if $n_0(t) > \frac{AT}{\tau}$	Probability of error can be obtained by evaluating the probability that $n_0(t) > \frac{AT}{\tau}$
$+A$	Error introduced if $n_0(t) < -\frac{AT}{\tau}$	Probability of error can be obtained by evaluating the probability that $n_0(t) < -\frac{AT}{\tau}$

Table: Probability of error in integrate and dump receiver

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2} \quad — (4)$$

where $f_X(x)$ is the PDF of random variable x

m is the mean value

σ is the standard deviation.

Here, we want to evaluate PDF for white gaussian noise we have,

$$x = n_0(t)$$

Since this noise has zero mean value, $m=0$ then eq (4) can be written as

$$f_X(n_0(t)) = \frac{1}{\sigma\sqrt{2\pi}} e^{-[n_0(t)]^2/2\sigma^2} \quad — (5)$$

The standard deviation σ is given as

$$\sigma = [\text{mean square value} - \text{square of mean value}]^{1/2}$$

$$= [\overline{X^2} - m_x^2]^{1/2}$$

mean square value $\overline{X^2} = \overline{n_0^2(t)} = \frac{N_0 T}{2\tau^2}$ from eq (13)

mean value $m_x = 0$ for this noise

$$\sigma = [\overline{n_0^2(t)}]^{1/2} = \sqrt{\frac{N_0 T}{2\tau^2}} \quad — (6)$$

Substitute eq (6) in eq (5), can be written as

$$f_X(n_0(t)) = \frac{1}{\sqrt{\frac{N_0T}{2\tau^2}} \cdot \sqrt{2\pi}} \cdot e^{-[n_0(t)]^2 / 2\left(\frac{N_0T}{2\tau^2}\right)}$$

$$= \frac{\tau}{\sqrt{\pi N_0 T}} e^{-[n_0(t)]^2 / \left(\frac{N_0T}{\tau^2}\right)} \quad -(7)$$

This equation gives PDF of white gaussian noise. Fig 5 shows the plot of this PDF.

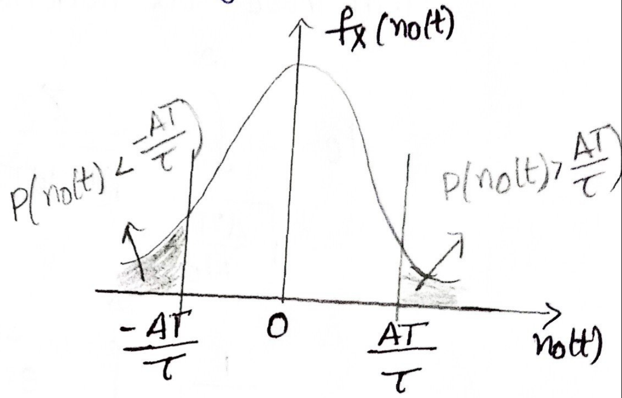


Fig 5: PDF of white gaussian noise of zero mean

From the property of PDF, we know that

$$P\left(n_0(t) > \frac{AT}{\tau}\right) = \int_{AT/\tau}^{\infty} f_X(n_0(t))\, d[n_0(t)] \quad -(8)$$

This equation gives the probability that $n_0(t)$ takes value greater than $\frac{AT}{\tau}$

Similarly the probability that $n_0(t)$ takes value less than $\frac{-AT}{\tau}$ is given by area under the curve from $-\frac{AT}{\tau}$ onwards on left side.

Since the PDF curve is symmetric we can write

$$P_e = P\left(n_0(t) > \frac{AT}{\tau}\right) = P\left(n_0(t) < \frac{-AT}{\tau}\right)$$

$$P_e = P\left(n_0(t) > \frac{AT}{\tau}\right) = \int_{AT/\tau}^{\infty} f_X(n_0(t))\, d[n_0(t)] \quad -(9)$$

Substitute the value of $f_X(n_0(t))$ in eq (9), we get

$$P_e = \int_{\frac{AT}{\tau}}^{\infty} \frac{\tau}{\sqrt{\pi N_0 T}} e^{-[n_0(t)]^2 / \left(\frac{N_0T}{\tau^2}\right)} \cdot d[n_0(t)] \quad -(10)$$

Put $\frac{[n_0(t)]^2}{\frac{N_0T}{\tau^2}} = y^2 \Rightarrow y = \frac{n_0(t)}{\sqrt{N_0T}/\tau}$

When $n_0(t) \to \infty \quad \Rightarrow y \to \infty$

When $n_0(t) \to \frac{AT}{\tau} \quad \Rightarrow y = \frac{AT/\tau}{\sqrt{N_0T}/\tau} = \sqrt{\frac{A^2T}{N_0}}$

With these substitutions eq (10) becomes

$$P_e = \int_{\sqrt{\frac{A^2T}{N_0}}}^{\infty} \frac{\tau}{\sqrt{\pi N_0 T}} e^{-y^2} \cdot \frac{\sqrt{N_0T}}{\tau} \cdot dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{A^2T}{N_0}}}^{\infty} e^{-y^2} \cdot dy \quad - (11)$$

Let us rearrange the above equation as follows:

$$P_e = \frac{1}{2} \left\{ \frac{2}{\sqrt{\pi}} \int_{\sqrt{\frac{A^2T}{N_0}}}^{\infty} e^{-y^2} \cdot dy \right\} \quad - (12)$$

The integration inside brackets can be evaluated with the help of Complementary error function i.e

$$\frac{2}{\sqrt{\pi}} \int_{u}^{\infty} e^{-y^2} \cdot dy = \text{erfc}(u) \quad - (13)$$

This is a standard result and normally evaluated using numerical methods. With the help of this definition the eq (12) becomes

$$P_e = \frac{1}{2} \text{erfc}\left(\sqrt{\frac{A^2T}{N_0}}\right) \quad - (14)$$

This equation gives probability of error of the integrate and dump receiver. Since $A^2T = E$ i.e energy of the bit we can write,

$$\boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\sqrt{\frac{E}{N_0}}}$$

Basically 'erfc' is the monotonically decreasing function, P_e falls rapidly as the ratio $\frac{E}{N_0}$ increases. Thus the maximum value of P_e is $\frac{1}{2}$ when $\frac{E}{N_0}$ is very very small. This means even if the signal is lost entirely in the noise, the probability of error will be $\frac{1}{2}$. That means the receiver will make incorrect decisions half number of times.

→ **OPTIMUM FILTER** (optimum Receiver)

A generalized filter is used to receive the binary coded signals and minimize the probability of error 'P_e' is called "optimum filter".

Let us consider the generalized Gaussian noise having zero mean.

Let us assume that the Bipolar NRZ signal is used to represent binary 1's and 0's i.e

For binary 1 $\quad x_1(t) = +A$ for one bit period

For binary 0 $\quad x_2(t) = -A$ for one bit period

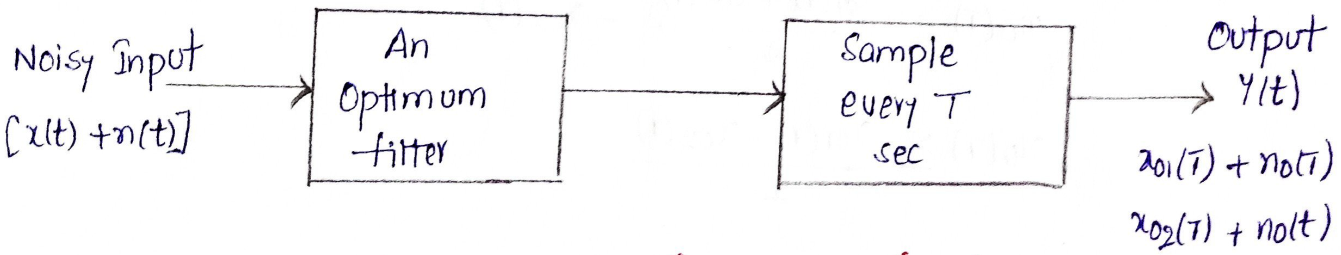


Fig 6 Block diagram of receiver for binary coded signal

As shown in figure 6, the noise $n(t)$ is added to the signal $x(t)$ in the channel during the transmission.

Thus, input to the optimum filter is $[x(t)+n(t)]$ and output from the receiver is $x_{01}(T) + n_0(T)$ or $x_{02}(T) + n_0(T)$

In the absence of noise $n(t)$, the output of the receiver will be

$$y(T) = x_{01}(T) \quad \text{if} \quad x(t) = x_1(t)$$

$$y(T) = x_{02}(T) \quad \text{if} \quad x(t) = x_2(t)$$

Hence, in the absence of noise, decisions are taken clearly. However, if noise is present then, we select $x_1(t)$ if $y(T)$ is closer to $x_{01}(T)$ than $x_{02}(T)$ and we select $x_2(t)$ if $y(T)$ is closer to $x_{02}(T)$ than $x_{01}(T)$

Therefore, the decision boundary will be in the middle of $x_{01}(T)$ and $x_{02}(T)$

It is expressed as

$$\text{Decision boundary} = \frac{x_{01}(T) + x_{02}(T)}{2}$$

→ **Calculation of Probability of Error (P_e) for optimum filter**

Let us consider that $x_2(t)$ was transmitted, but $x_0(T)$ is greater than $x_{02}(T)$. If noise $n_0(T)$ is positive and larger in magnitude compared to the voltage difference $\frac{x_{01}(T) + x_{02}(T)}{2} - x_{02}(T)$ then in this case the incorrect decision will be taken. This means that the error will be generated if

$$n_0(T) \geq \frac{x_{01}(T) + x_{02}(T)}{2} - x_{02}(T)$$

$$n_0(T) \geq \frac{x_{01}(T) - x_{02}(T)}{2}$$

We have obtained the probability density function (PDF) for $n_0(t)$ in the Integrate and Dump filter receiver is

$$f_x[n_0(t)] = \frac{1}{\sigma\sqrt{2\pi}} e^{-[n_0(t)]^2/2\sigma^2} \quad -①$$

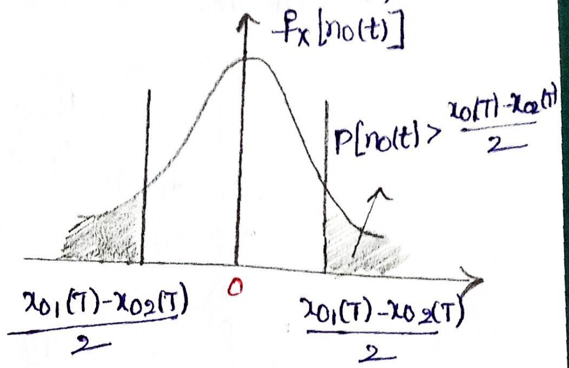


Fig 7: Evaluation of P_e for optimum filter receiver

where $n_0(t)$ is the random function whose PDF is given by above equation

σ is the standard deviation

Hence, to evaluate the probability of error, we must integrate the area under the PDF curve from $n_0(t) \geq \frac{x_{01}(T)-x_{02}(T)}{2}$. This portion of the curve has been shown shaded in figure

Therefore, we have

$$P_e = P\left[n_0(t) \geq \frac{x_{01}(T)-x_{02}(T)}{2}\right] = \int_{\frac{x_{01}(T)-x_{02}(T)}{2}}^{\infty} f_x[n_0(t)]\, d[n_0(t)] \quad -⑤$$

Substitute the eq ① in eq ⑤, then we get

$$P_e = \int_{\frac{x_{01}(T)-x_{02}(T)}{2}}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-[n_0(t)]^2/2\sigma^2} \cdot d[n_0(t)]$$

Let $\frac{[n_0(t)]^2}{2\sigma^2} = y^2$

$$y = \frac{n_0(t)}{\sqrt{2}\sigma}$$

$$dy = \frac{d[n_0(t)]}{\sqrt{2}\sigma}$$

$$d[n_0(t)] = \sqrt{2}\sigma\, dy$$

Limits: when $n_0(t) = \frac{x_{01}(T)-x_{02}(T)}{2}$ then

$$y = \frac{n_0(t)}{\sqrt{2}\sigma}$$

$$= \frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\sigma}$$

Substituting all these values, we get

$$P_e = \int_{\frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\sigma}}^{\infty} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-y^2} \cdot \cancel{\sigma}\sqrt{2}\, dy \quad - (3)$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\sigma}}^{\infty} e^{-y^2} \cdot dy$$

Let us rearrange this equation as under

$$P_e = \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int_{\frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\sigma}}^{\infty} e^{-y^2} dy\right] \quad - (4)$$

To solve this integration, let us use the following standard result

$$= \frac{2}{\sqrt{\pi}} \int_{u}^{\infty} e^{-y^2} dy = \operatorname{erfc}(u) \quad - (5)$$

Substitute the above equation in eq (4), we get

$$P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\sigma}\right]$$

This is the required expression for error probability P_e of optimum filter. It may be noted that the 'erfc' function is the monotonically decreasing function. Hence, P_e decreases as the difference $x_{01}(T) - x_{02}(T)$ becomes greater and the rms noise voltage 'σ' becomes smaller. The optimum filter has to maximize the ratio $\frac{x_{01}(T) - x_{02}(T)}{\sigma}$ in such a manner that the probability of error P_e is maximum.

→ Evaluation of Transfer Function for the Optimum Filter

Let us find the transfer function of the optimum filter in such a way that it will maximize the ratio $\frac{x_{01}(T) - x_{02}(T)}{\sigma}$. For this, Let us represent the difference signal $x_{01}(T) - x_{02}(T)$ by $x_0(T)$. Thus, we have

$$x_0(T) = x_{01}(T) - x_{02}(T) \quad — (1)$$

This means that the optimum filter has to maximize the ratio $\frac{x_0(T)}{\sigma}$.

Now, Let us derive the transfer function of the optimum filter such that the square of the ratio $\frac{x_0(T)}{\sigma}$ is maximized.

The square of this ratio is

$$\left(\frac{S}{N}\right)_0 = \frac{x_0^2(T)}{\sigma^2} \quad — (2)$$

where $\left(\frac{S}{N}\right)_0$ is known as the signal to noise power ratio.

Further, $\sigma^2 = \overline{n_0^2(T)} = E[n_0^2(T)]$ is normalized noise power because mean value of noise is zero. Thus, the optimum filter has to maximize the ratio

$$\left(\frac{S}{N}\right)_0 = \frac{x_0^2(T)}{\overline{n_0^2(T)}} = \frac{x_0^2(T)}{E[n_0^2(T)]} = \frac{x_0^2(T)}{\sigma^2} \quad — (3)$$

If $X(f)$ is the Fourier transform of input difference signal $x(t)$ i.e $[x(t) = x_1(t) - x_2(t)]$, then we have

$$X_0(f) = H(f)X(f) \quad — (4)$$

where $H(f)$ is the transfer function of optimum filter

$x_0(T)$ is found by taking inverse Fourier transform of $X_0(f)$ ie

$$x_0(T) = IFT[X_0(f)] = \int_{-\infty}^{\infty} X_0(f) e^{j2\pi fT} df \quad — (5)$$

$$x_0(T) = \int_{-\infty}^{\infty} H(f)\,X(f)\,e^{j2\pi fT}\,df \quad -⑥$$

Also, the input noise to the optimum filter is $n(t)$. Let its power spectral density (psd) be $S_{ni}(f)$. The output noise of optimum filter is $n_0(t)$. Let its power spectral density be $S_{no}(f)$.

The relation between input and output power spectral densities are

$$S_{no}(f) = |H(f)|^2 S_{ni}(f) \quad -⑦$$

The normalized noise power can be obtained by integrating the power spectral density

$$\sigma^2 = \int_{-\infty}^{\infty} S_{no}(f)\,df = \int_{-\infty}^{\infty} |H(f)|^2 S_{ni}(f) \quad -⑧$$

Substitute the eq ⑥ and ⑧ in eq ③, then we get

$$\left(\frac{S}{N}\right)_0 = \frac{x_0^2(T)}{\sigma^2} = \frac{\left|\int_{-\infty}^{\infty} H(f)\,X(f)\,e^{j2\pi fT}\,df\right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 S_{ni}(f)\,df} \quad -⑨$$

The schwarz's inequality states that

$$\left|\int_{-\infty}^{\infty} \theta_1(x)\,\theta_2(x)\,dx\right|^2 \le \int_{-\infty}^{\infty} |\theta_1(x)|^2 dx \int_{-\infty}^{\infty} |\theta_2(x)|^2 dx \quad -⑩$$

Let us consider that $\theta_1(f) = \sqrt{S_{ni}(f)}\,H(f)$

$$\theta_2(f) = \frac{1}{\sqrt{S_{ni}(f)}}\,X(f)\,e^{j2\pi fT}$$

Apply the schwarz's inequality for eq ⑨, then we get

$$\left(\frac{S}{N}\right)_0 \le \frac{\int_{-\infty}^{\infty} |\theta_1(f)|^2 df \int_{-\infty}^{\infty} |\theta_2(f)|^2 df}{\int_{-\infty}^{\infty} |\theta(f)|^2 df} \quad -⑪$$

Substituting the values $\theta_1(f)$ and $\theta_2(f)$ in eq (11) then we get

$$\left(\frac{S}{N}\right)_0 \leq \frac{\int_{-\infty}^{\infty} \cancel{|S_{ni}(f) H^2(f)|\, df} \cdot \int_{-\infty}^{\infty} \frac{1}{S_{ni}(f)} |X(f) e^{j2\pi fT}|^2 df}{\cancel{\int_{-\infty}^{\infty} |H(f)|^2 S_{ni}(f)\, df}}$$

$$\left(\frac{S}{N}\right)_0 \leq \int_{-\infty}^{\infty} \frac{1}{S_{ni}(f)} |X(f) e^{j2\pi fT}|^2 df$$

$$\leq \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)} df \qquad \text{where} \left[\because |X(f) e^{j2\pi fT}|^2 = |X(f)|^2 \quad |e^{j2\pi fT}| = 1\right]$$

The signal to Noise power ratio will be maximum when we consider equality

$$\boxed{\therefore \left(\frac{S}{N}\right)_{0\,max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)} df} \quad \text{— (12)}$$

The equality is possible only if we have $\theta_1(f) = K\theta_2^*(f)$

substituting the values of $\theta_1(f)$ and $\theta_2(f)$, We obtain

$$\theta_1(f) = K \cdot \theta_2^*(f)$$

$$\sqrt{S_{ni}(f)}\, H(f) = K \cdot \frac{1}{\sqrt{S_{ni}(f)}} X^*(f) e^{-j2\pi fT}$$

$$\boxed{H(f) = K \cdot \frac{X^*(f)}{S_{ni}(f)} e^{-j2\pi fT}} \quad \text{— (13)}$$

The optimum filter which minimizes the probability of error (Pe) has to maximize the ratio $\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2$. To maximize this ratio, the filter has the transfer function given by following expression,

$$H(f) = K \cdot \frac{X^*(f)}{S_{ni}(f)} e^{-j2\pi fT}$$

MATCHED FILTER

The mathematical operation of a matched filter (MF) is convolution. i.e a signal is convolved with the impulse response of a filter.

In optimum filter, we considered the generalized Gaussian Noise. When this noise is white Gaussian noise, the optimum filter is known as "Matched filter".

Calculation of Impulse Response for the Matched Filter:

The transfer function of the optimum filter is

$$H(f) = k \cdot \frac{X^*(f)}{S_{ni}(f)} e^{-j2\pi fT} \quad \text{---} (1)$$

We know that $S_{ni}(f) = \frac{N_0}{2}$ — (2)

substitute the eq (2) in eq (1) then we get

$$H(f) = k \cdot \frac{X^*(f)}{\frac{N_0}{2}} e^{-j2\pi fT}$$

$$= \frac{2k}{N_0} X^*(f) e^{-j2\pi fT} \quad \text{---} (3)$$

From the property of Fourier transform, We know that

$$X^*(f) = X(-f) \quad \text{---} (4)$$

Using this property, we can write the eq (3) as under:

$$H(f) = \frac{2k}{N_0} X(-f) e^{-j2\pi fT} \quad \text{---} (5)$$

The impulse response of a matched filter can be evaluated by taking inverse Fourier transform of above equation

$$h(t) = IFT[H(f)]$$

$$h(t) = \text{IFT}\left[\frac{2K}{N_0} X(-f) e^{-j2\pi fT}\right] \quad -⑥$$

The inverse Fourier transform of $X(-f)$ is $x(-t)$ and $e^{-j2\pi fT}$ represents time shift of 'T' seconds.

Hence, We have

$$FT[x(-t)] = X(-f)$$

$$FT[x(T-t)] = X(-f)e^{-j2\pi fT}$$

With the help of all these properties of Fourier transform, the eq ⑥ will become

$$\boxed{h(t) = \frac{2K}{N_0} x(T-t)} \quad -⑦$$

We know that $x(t) = x_1(t) - x_2(t)$, therefore the eq ⑦ will become

$$\boxed{h(t) = \frac{2K}{N_0}\left[x_1(T-t) - x_2(T-t)\right]} \quad -⑧$$

The equation ⑦ & ⑧ gives the required impulse response of the matched filter.

→ Calculation of Probability of Error (P_e) for the Matched Filter

To evaluate the probability of error for matched filter, we consider the probability of error for optimum filter.

We know that error probability of optimum filter is expressed as

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right] \quad -①$$

The transfer function of optimum filter is

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)}\, df \quad -②$$

We know that power spectral density (psd) of White noise is

$$S_{ni}(f) = \frac{N_0}{2} \quad -③$$

Substitute the eq ③ in eq ②, we get

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{\frac{N_0}{2}} df = \frac{2}{N_0}\int_{-\infty}^{\infty} |X(f)|^2 df \quad -④$$

The Parseval's power theorem states that,

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{0}^{T} x^2(t)\,dt \quad -⑤$$

In the last integral we have taken limits from 0 to T because $x(t)$ exists from 0 to T only. We know that $x(t) = x_1(t) - x_2(t)$

Hence, the eq ⑤ becomes

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{0}^{T} [x_1(t) - x_2(t)]^2 dt$$

$$= \int_{0}^{T} x_1^2(t) + x_2^2(t) - 2x_1(t)x_2(t)\,dt$$

$$= \int_{0}^{T} x_1^2(t)\,dt + \int_{0}^{T} x_2^2(t)\,dt - 2\int_{0}^{T} x_1(t)x_2(t)\,dt \quad -⑥$$

where $\int_{0}^{T} x_1^2(t)dt = E_1$ i.e energy of $x_1(t)$ by standard relations.

$\int_{0}^{T} x_2^2(t)dt = E_2$ and $\int_{0}^{T} x_1(t)x_2(t)\,dt = E_{12}$ represents energy due to autocorrelation between $x_1(t)$ and $x_2(t)$.

Now, if we choose $x_1(t) = -x_2(t)$, then these energies will be equal i.e

$$E_1 = E_2 = -E_{12} = E$$

Substituting all these values in eq ⑥, We get

$$\int_{-\infty}^{\infty} |X(f)|^2 df = E + E - 2(-E) = 4E \quad -⑦$$

Substituting the value in eq ④, then we get

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0} \cdot 4E = \frac{8E}{N_0}$$

$$\frac{x_{01}(T) - x_{02}(T)}{\sigma} = 2\sqrt{2}\sqrt{\frac{E}{N_0}} \quad -⑧$$

Substituting the eq ⑧ value in eq ①, then we get the probability of error of matched filter as under

$$P_e = \frac{1}{2} \text{erfc}\left[\frac{2\sqrt{2}\sqrt{E/N_0}}{2\sqrt{2}}\right]$$

$$= \frac{1}{2} \text{erfc}\left[\sqrt{\frac{E}{N_0}}\right]$$

The minimum error probability of Matched filter is

$$\boxed{P_e = \frac{1}{2} \text{erfc}\sqrt{\frac{E}{N_0}}}$$

<u>Few points about Error probability (P_e) of Matched Filter</u>.

1. The error probability depends only upon the signal Energy E. It does not depend on the shape of the signal.

2. The probability of error of matched filter is equivalent to Integrate and dump filter when $x_1(t) = -x_2(t) = A$;

In other words, we can say that for a rectangular bipolar pulse input, the integrate and dump filter is same as a "Matched filter".

THE CORRELATOR : COHERENT RECEPTION

The mathematical operation of a correlator is correlation i.e a signal is correlated with a replica of itself. Infact, the term "Matched-filter" is often used synonymously with "correlator"

It is important to note that the correlator output and the Matched filter output are the same only at time $t=T$

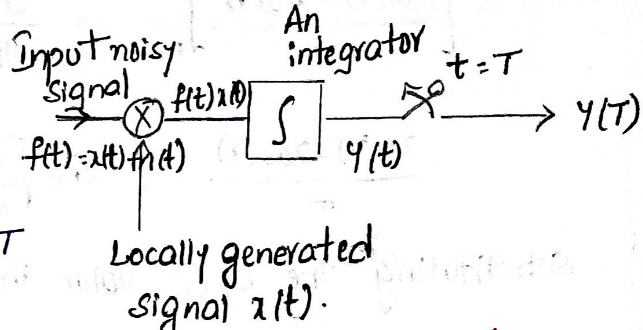


Fig 8: Block diagram of a correlator

Let us consider a little different type of receiver which is known as correlator. The Figure 8 shows the block diagram of this correlator. Let $f(t)$ represents input noisy signal,

$$f(t) = x(t) + n(t)$$

The signal $f(t)$ is multiplied to the locally generated replica of input signal $x(t)$. Then the result of multiplication $f(t) \cdot x(t)$ is integrated. The output of the integrator is sampled at $t=T$ (i.e end of one symbol period). Then based on this sampled value, decision is made. This is how the correlator works.

The output $y(t)$ will be $\quad y(t) = \int_0^T f(t)\,x(t)\,dt \quad$ — ①

Now consider the Matched filter as shown in fig 9: In this block diagram, it may be observed that the matched filter does not need locally generated replica of input signal $x(t)$.

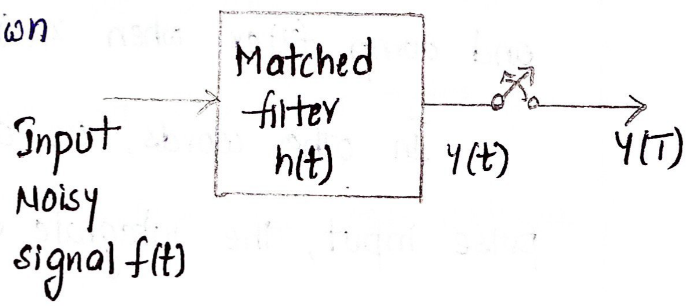


Fig 9: Block diagram of a Matched filter

The output of the matched filter is obtained by convolution of input $f(t)$ and its impulse response $h(t)$

This means that

$$y(t) = f(t) \circledast h(t) = \int_{-\infty}^{\infty} f(\tau) \cdot h(t-\tau)\, d\tau$$

We know that the impulse response $h(t)$ of the matched filter is given by,

$$h(t) = \frac{2k}{N_0} x(T-t)$$

$$h(t-\tau) = \frac{2k}{N_0} x(T-t+\tau) = \frac{2k}{N_0} x(T)$$

Because the integration is performed over one bit period, therefore, we can change integration limits from 0 to T.

Hence, we write

$$y(t) = \frac{2k}{N_0} \int_{0}^{T} f(\tau)\, x(T-t+\tau)\, d\tau$$

At $t = T$, the last equation becomes

$$y(T) = \frac{2k}{N_0} \int_{0}^{T} f(\tau)\, x(T-T+\tau)\, d\tau = \frac{2k}{N_0} \int_{0}^{T} f(\tau)\, x(\tau)\, d\tau$$

Now, let us substitute $\tau = t$ just for convenience for notation, then we have output of Matched filter

$$y(T) = \frac{2k}{N_0} \int_{0}^{T} f(t)\, x(t)\, dt \quad \text{--- (2)}$$

It may be observed that this eq (1) & (2) are identical. In eq (2) the constant $\frac{2k}{N_0}$ is present which can be normalized to 1. The Matched filter and correlator provides same output. Infact, these two techniques are used to synthesize the optimum filter.

ERROR PROBABILITY OF ASK

We know that the ASK signal is represented as

Binary 1 : $x_1(t) = \sqrt{2P_s}\cos\omega_c t$

Binary 0 : $x_2(t) = 0$

where P_s = Average normalized signal power

$$P_s = \frac{A^2}{2}$$

We know that the error probability with optimum filter is

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right] \quad -①$$

We also derived the expression for maximum output signal to noise ratio of an optimum filter as

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)}\,df \quad -②$$

The value of psd of white noise input is $S_{ni}(f) = N_0/2$.

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_{-\infty}^{\infty} |X(f)|^2\,df \quad -③$$

According to the Rayleigh's energy theorem,

$$\int_{-\infty}^{\infty} |X(f)|^2\,df = E = \int_{-\infty}^{\infty} x^2(t)\,dt \quad -④$$

But as the signal $x(t)$ is present only over a bit interval 'T', the limits of integration will change from 0 to T. The eq ③ can be written as

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T x^2(t)\,dt \quad -⑤$$

where $x(t) = x_1(t) - x_2(t)$

But in case of ASK $x_2(t) = 0$. Therefore $x_1(t) = \sqrt{2P_s}\cos\omega_c t$

Substitute the value of $x(t)$ in eq ⑤, then

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T 2P_s \cos^2\omega_c t\, dt$$

$$= \frac{4P_s}{N_0}\int_0^T \cos^2\omega_c t\, dt = \frac{4P_s}{N_0}\int_0^T \left(\frac{1+\cos 2\omega_c t}{2}\right) dt$$

$$= \frac{2P_s}{N_0}\int_0^T (1+\cos 2\omega_c t)\, dt$$

$$= \frac{2P_s}{N_0}\left[\int_0^T 1.\, dt + \int_0^T \cos 2\omega_c t\, dt\right]$$

$$= \frac{2P_s}{N_0}\left\{[t]_0^T + \frac{1}{2\omega_c}[\sin 2\omega_c t]_0^T\right\}$$

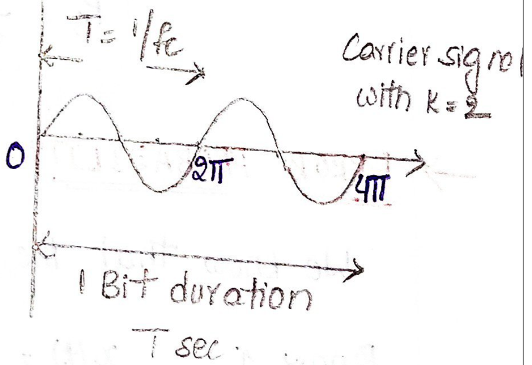
$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2P_s}{N_0}\left\{T + \frac{\sin 2\omega_c T}{2\omega_c}\right\} \quad —⑥$$

Let us consider the second term on the right hand side of the equation ⑥ i.e

$$\sin 2\omega_c T = \sin(2\times 2\times \pi f_c T)$$

$$= \sin 4\pi f_c T \quad —⑦$$

We assume that the frequency of the carrier signal (f_c) is selected such that there are 'k' no. of complete cycles of the carrier during one bit duration 'T' as shown in figure.



One bit period $T = 2T_c = \frac{2}{f_c}$

$$f_c T = 2 \quad —⑧$$

Here, $k = 2$, therefore eq ⑦ can be written in general as under.

$$f_c T = k$$

In eq ⑦ $\sin 2\omega_c T = \sin 4\pi k$ where $k = 1, 2, 3 \ldots$

$= \sin 4\pi, \sin 8\pi, \sin 12\pi \ldots$ etc for $k = 1, 2, 3 \ldots$

$\sin 2\omega_c T = 0$ for all values of k.

Therefore eq ⑥ gets modified as under

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2P_sT}{N_0}$$

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]_{max} = \sqrt{\frac{2P_sT}{N_0}} \quad — ⑨$$

Let us substitute the eq ⑨ in eq ① to obtain the error probability for ASK

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{\sqrt{\frac{2P_sT}{N_0}}}{2\sqrt{2}}\right] = \frac{1}{2}\,\text{erfc}\left[\frac{1}{2}\sqrt{\frac{2P_sT}{N_0}}\right]$$

$$\boxed{P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{P_sT}{4N_0}}}$$

But $P_sT = E$ = Energy of the signal

The bit error probability denoted by P_B

$$\boxed{P_B = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E}{4N_0}}}$$

→ **ERROR PROBABILITY OF BPSK (COHERENT DETECTION)**

We know that the BPSK signal is represented as follows:

Binary 1 $\quad x_1(t) = \sqrt{2P_s}\cos\omega_c t \quad$ — ①

Binary 0 $\quad x_2(t) = -\sqrt{2P_s}\cos\omega_c t \quad$ — ②

$\therefore \quad x_2(t) = -x_1(t) \quad$ — ③

We have to use the matched filter for detection of BPSK signal. The expression for error probability of an optimum filter is given by

$$P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\sigma}\right] \quad - (4)$$

The expression for the signal to noise ratio of a Matched filter is

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0} \int_{-\infty}^{\infty} |X(f)|^2 df \quad - (5)$$

using the Rayleigh's energy theorem, We have

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_0^T x^2(t)\, dt \quad - (6)$$

Substitute the eq (6) in eq (5), then we get

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0} \int_0^T x^2(t)\, dt \quad - (7)$$

But $x(t) = x_1(t) - x_2(t)$

For BPSK $= x_2(t) = -x_1(t)$ [from eq (3)]

$$\therefore x(t) = 2x_1(t)$$

$$= 2\sqrt{2P_s} \cos\omega_c t \quad - (8)$$

substituting eq (8) in eq (7), then we get

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0} \int_0^T 8P_s \cos^2\omega_c t\, dt$$

$$= \frac{16 P_s}{N_0} \int_0^T \cos^2\omega_c t\, dt$$

$$= \frac{16 P_s}{N_0} \int_0^T \left(\frac{1 + \cos 2\omega_c t}{2}\right) dt$$

$$= \frac{8P_s}{N_0} \int_0^T (1 + \cos 2\omega_c t)\, dt$$

$$= \frac{8P_s}{N_0} \left[\int_0^T 1 . dt + \int_0^T \cos 2\omega_c t\, dt\right]$$

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{8P_s}{N_0}\left[(t)_0^T + \frac{1}{2\omega_c}(\sin 2\omega_c t)_0^T \right]$$

$$= \frac{8P_s}{N_0}\left[T + \frac{\sin 2\omega_c T}{2\omega_c}\right] \quad \text{—} (9)$$

The second term in the RHS of eq (9) is zero

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{8P_sT}{N_0}$$

$$= \frac{8E}{N_0} \qquad [P_sT = E]$$

$$\frac{x_{01}(T)-x_{02}(T)}{\sigma} = \sqrt{\frac{8E}{N_0}} \quad \text{—} (10)$$

Substitute the eq (10) in eq (4) then we get the error probability of BPSK as

$$P_e = \frac{1}{2}\operatorname{erfc}\left[\frac{1}{2\sqrt{2}}\cdot\sqrt{\frac{8E}{N_0}}\right]$$

$$\boxed{P_B = \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{E}{N_0}}\right]}$$

→ **BIT ERROR PROBABILITY OF COHERENTLY DETECTED BFSK**

Let us assume that a matched filter is used for the detection of BFSK signal. In BFSK, the received signal is as follows:

For Binary 1 : $x_1(t) = A\cos(\omega_c+\Omega)t$ — (1)

Binary 0 : $x_2(t) = A\cos(\omega_c-\Omega)t$ — (2)

We know that one way of synthesizing a matched filter is to construct a correlation receiver system, because the correlation receiver will give exactly the same performance as a Matched filter provided that the locally generated waveform is $[x_1(t) - x_2(t)]$

Therefore, local signal : $x_1(t) - x_2(t)$

$$x(t) = A\cos(\omega_c+\Omega)t - A\cos(\omega_c-\Omega)t \quad — (3)$$

where $A = \sqrt{2P_s}$

$$\therefore x_1(t) - x_2(t) = \sqrt{2P_s}\left[\cos(\omega_c+\Omega)t - \cos(\omega_c-\Omega)t\right] \quad — (4)$$

We know that the error probability of a Matched filter is given by

$$P_e = \frac{1}{2}\operatorname{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\sigma}\right] \quad — (5)$$

We know that the maximum signal to Noise ratio is given by

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T x^2(t)\,dt \quad — (6)$$

substitute the eq (3) in eq (6) then we get

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T \left\{\sqrt{2P_s}\left[\cos(\omega_c+\Omega)t - \cos(\omega_c-\Omega)t\right]\right\}^2 dt \quad — (7)$$

To obtain the value of $x^2(t)$, we have

$$x^2(t) = \left\{\sqrt{2P_s}\left[\cos(\omega_c+\Omega)t - \cos(\omega_c-\Omega)t\right]\right\}^2$$

$$= 2P_s\left[\cos(\omega_c+\Omega)t - \cos(\omega_c-\Omega)t\right]^2$$

We know that $\cos(A+B) - \cos(A-B) = -2\sin A\sin B$

$$\therefore x^2(t) = 2P_s\left[-2\sin(\omega_c t)\sin(\Omega t)\right]^2$$

$$= 2P_s\left[4\sin^2(\omega_c t)\sin^2(\Omega t)\right]$$

$$= 2P_s\left[2\sin^2(\omega_c t)\cdot 2\sin^2(\Omega t)\right]$$

$$= 2P_s\left[(1-\cos 2\omega_c t)(1-\cos 2\Omega t)\right]$$

$$= 2P_s\left[1 - \cos 2\Omega t - \cos 2\omega_c t + \cos 2\omega_c t\cos 2\Omega t\right]$$

$$x^2(t) = 2P_s\left[1 - \cos 2\Omega t - \cos 2\omega_c t + \frac{1}{2}\cos 2(\omega_c-\Omega)t + \frac{1}{2}\cos 2(\omega_c+\Omega)t\right] \quad — (8)$$

$$\left[\because \cos A\cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]\right]$$

Substitute the value of eq ⑧ in eq ⑥ then we get

Taking integration of the both sides, We get

$$\int_0^T x^2(t)dt = \int_0^T 2P_s \{1 - \cos 2\Omega t - \cos 2\omega_c t + \frac{1}{2}[\cos 2(\omega_c - \Omega)t + \frac{1}{2}\cos 2(\omega_c+\Omega)t\} dt$$

$$= 2P_s \{ T - \frac{\sin 2\Omega T}{2\Omega} - \frac{\sin 2\omega_c T}{2\omega_c} + \frac{1}{2}\frac{\sin 2(\omega_c - \Omega)T}{2(\omega_c-\Omega)} + \frac{1}{2}\cdot\frac{\sin 2(\omega_c+\Omega)T}{2(\omega_c+\Omega)}$$

$$\int_0^T x^2(t)dt = 2P_sT \{ 1 - \frac{\sin 2\Omega T}{2\Omega T} - \frac{\sin 2\omega_c T}{2\omega_c T} + \frac{\sin 2(\omega_c-\Omega)T}{4(\omega_c-\Omega)} + \frac{\sin(\omega_c+\Omega)T}{4(\omega_c+\Omega)} \} \quad — ⑨$$

If we assume that the offset angular frequency 'Ω' is very small as compared to the carrier angular frequency ω_c, then the last three terms will have a form $\sin(2\omega_c T)/2\omega_c T$. This ratio approaches zero as the value of $\omega_c t$ increases.

Generally, $\omega_c T >> 1$ therefore the last three terms in RHS of the above equation can be neglected. Therefore, the eq ⑨ gets modified to

$$\int_0^T x^2(t)\,dt = 2P_sT\left[1 - \frac{\sin 2\Omega T}{2\Omega T}\right] \quad — ⑩$$

substitute the eq ⑩ in eq ⑧, We get

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\cdot 2P_sT\left[1 - \frac{\sin 2\Omega T}{2\Omega T}\right]$$

$$= \frac{4P_sT}{N_0}\left[1 - \frac{\sin 2\Omega T}{2\Omega T}\right] \quad — ⑪$$

The value of the quantity $\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max}$ attains its largest value when $2\Omega T = \frac{3\pi}{2}$. substituting this value in eq ⑪, We get

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{4P_sT}{N_0}\left[1-\frac{\sin(3\pi/2)}{(3\pi/2)}\right]$$

$$= \frac{4P_sT}{N_0}\left[1-\frac{(-1)}{(3\pi/2)}\right] = \frac{4P_sT}{N_0}\left[1+\frac{1}{3\pi/2}\right]$$

$$= 4.84\frac{P_sT}{N_0}$$

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]_{max} = \sqrt{\frac{4.84P_sT}{N_0}} \quad \text{--- (12)}$$

Substituting the eq (12) in eq (5) then the error probability as under:

$$P_e = \frac{1}{2}\operatorname{erfc}\left[\frac{1}{2\sqrt{2}}\cdot\sqrt{\frac{4.84P_sT}{N_0}}\right]$$

$$= \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{0.6P_sT}{N_0}}\right]$$

$$\boxed{P_e = \frac{1}{2}\operatorname{erfc}\left[\sqrt{\frac{0.6E}{N_0}}\right]} \qquad \because P_sT = E$$

As the erfc is a monotonically decreasing function, the error probability for BFSK system is higher than that of a BPSK system. This happens because in BPSK, $x_2(t) = -x_1(t)$ and in BFSK this condition is not satisfied.

→ BIT ERROR PROBABILITY FOR NON-COHERENTLY DETECTED BINARY FSK

Let us consider the BFSK system in which the non-coherent detection using the BPF is done. The bit error probability of such a system is given by

$$P_B = \frac{1}{2}\exp\left[\frac{-A^2}{4N_0 W_f}\right]$$

where A = Peak signal amplitude

$N_0/2$ = PSD of white noise ; W_f = filter Bandwidth

The above equation states that the error performance of the Non-coherent BFSK system is dependent on the bandwidth of the filter "W_f". The error probability decreases with decreases in the filter bandwidth W_f. This expression is valid only when the intersymbol interference (ISI) is negligible.

Probability of Error for QPSK

The signal space representation of QPSK is shown in fig 11. In the figure, observe that transmitted reference carriers are $\phi_1(t)$ and $\phi_2(t)$. All the signal vectors A, B, C and D are at 45° to these reference carriers. Consider that signal vector 'A' is transmitted. If phase shift of the reference carrier is more than 45°, it will be detected as 'B' (or) 'D'. It will depend upon phase shift of $\phi_2(t)$ also. Therefore consider the receiver for QPSK signal. There are two correlators for two reference carriers. These two correlators are actually BPSK receivers. Error probability of BPSK, due to imperfect phase is given by equation.

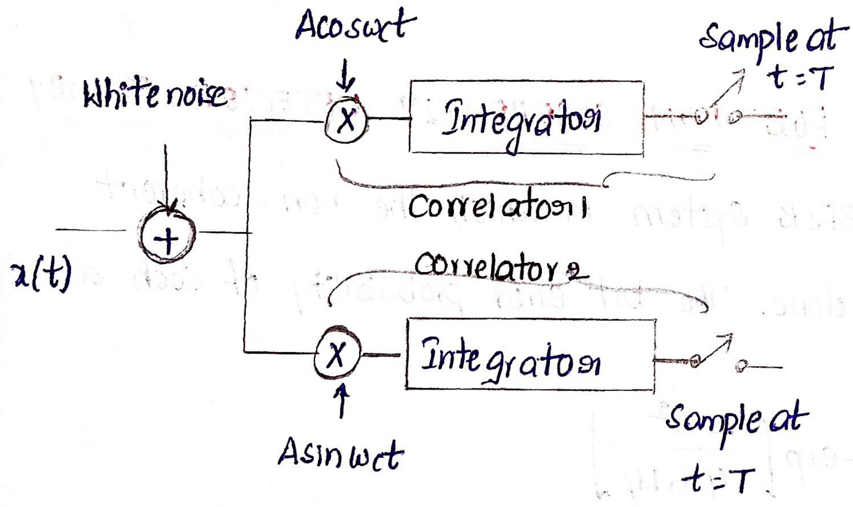


Fig 10: A correlation receiver for QPSK

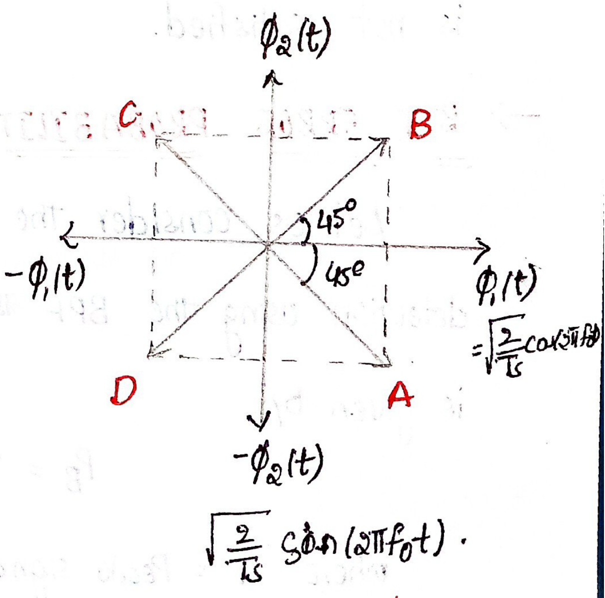


Fig 11: signal space diagram of QPSK

$$P_e = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{N_0} \cos^2\phi} \quad -①$$

Hence error probability of correlator I is given as,

$$P_{e_1} = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{N_0} \cos^2\phi} \quad -②$$

Since both the correlators of BPSK, the error probability of correlator 2 will be same as correlator I i.e

$$P_{e_2} = P_{e_1} = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{N_0} \cos^2\phi} \quad -③$$

From fig 11: observe that correlators detect wrong symbol if phase shift of the carrier is more than 45°. Hence putting $\phi = 45°$ in above equation

$$P_{e_1} = P_{e_2} = \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{N_0} \cos^2(45)}$$

$$= \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{N_0} \left(\frac{1}{\sqrt{2}}\right)^2} \quad \left[\because \cos 45° = 1/\sqrt{2}\right)$$

$$= \frac{1}{2} \operatorname{erfc} \sqrt{\frac{E}{2N_0}} \quad -④$$

Hence probability of getting correct symbol can be expressed as

$$P_c = (1-P_{e_1})(1-P_{e_2})$$

$$= 1 - P_{e_1} - P_{e_2} - P_{e_1} \cdot P_{e_2} \quad -⑤$$

We know that $P_{e_1} = P_{e_2}$ then the above equation becomes

$$P_c = 1 - 2P_{e_1} + (P_{e_1})^2 \quad -⑥$$

Normally P_{e_1} is very very small ($<<1$). Hence $P_{e_1}^2$ will be negligible i.e

$$P_c = 1 - 2P_{e_1} \quad -⑦$$

Probability of error is given in terms of 'P_c' as

$$P_e = 1 - P_c$$

$$P_e = 1 - P_c$$
$$= 1 - (1 - 2P_{e_1})$$
$$P_e = 2P_{e_1}. \quad - (8)$$

substitute the eq (4) value in eq (8), then we get

$$P_e = 2 \times \frac{1}{2} \, \text{erfc} \sqrt{\frac{E}{2N_0}}$$

$$\boxed{P_e = \text{erfc} \sqrt{\frac{E_b}{2N_0}}}$$

Formulas

1.	Signal to Noise ratio of Integrate and Dump filter	$\left(\frac{S}{N}\right)_0 = \frac{A^2T}{(N_0/2)}$
2.	Probability of error of Dump receiver	$P_e = \frac{1}{2}\,\text{erfc}\sqrt{E/N_0}.$
3.	Probability of error of optimum filter	$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\sigma}\right]$
4.	Probability of error of Matched filter	$P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E}{N_0}}$
5.	Transfer function of optimum filter	$H(f) = \int_{-\infty}^{\infty} \frac{\lvert X(f)\rvert^2}{S_{ni}(f)}$
6.	Probability of error for ASK	$P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E}{4N_0}}$
7.	Probability of error for PSK	$P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E}{N_0}}$
8.	Probability of error for BFSK (coherent)	$P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{0.6E}{N_0}}$
9.	Probability of error for BFSK (Non-coherent)	$P_e = \frac{1}{2}\exp\left[-A^2/4N_0w_f\right]$
10.	Probability of error for QPSK (coherent)	$P_e = \text{erfc}\sqrt{\frac{E}{N_0}}$